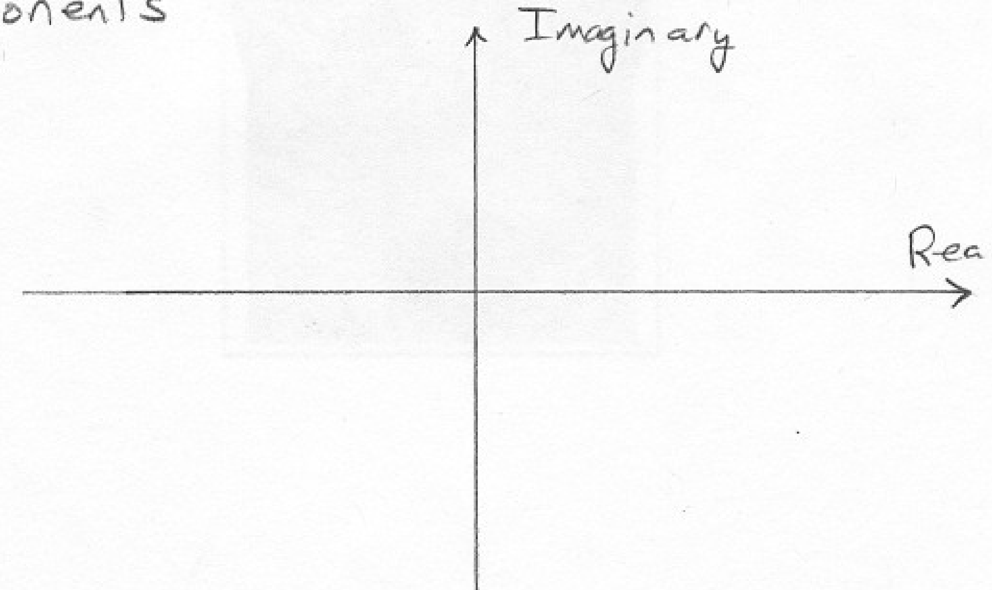
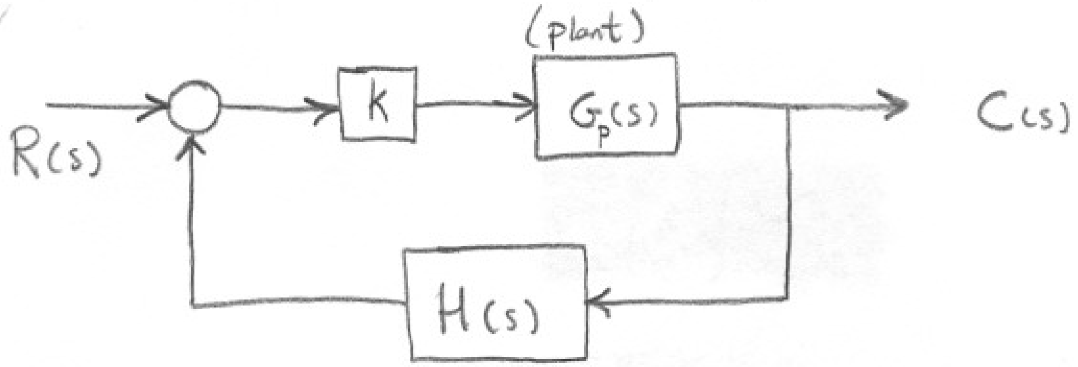
Root Locus Analysis

- A root locus plot is a way to graph the location of the closed loop poles as a compensator parameter is varied. It provides us with a qualitative way to observe the system performance. It is especially useful for systems of order higher than two. It also gives a graphic representation of a system's stability.
- For open loop systems the poles typically do not change with changes in the system gain. This is not true for closed loop systems.
- The poles of a closed loop transfer function are more difficult to find and they change with changes in the system gain. The closed loop poles are typically found by factoring the closed loop characteristic polynomial.

- The root locus is plotted on the S-Plane that consists of real (x-axis) and imaginary (y-axis) components



Consider the following system



where:
$$G_p(s) = \frac{s+1}{s(s+2)} \qquad H(s) = \frac{s+3}{s+4}$$

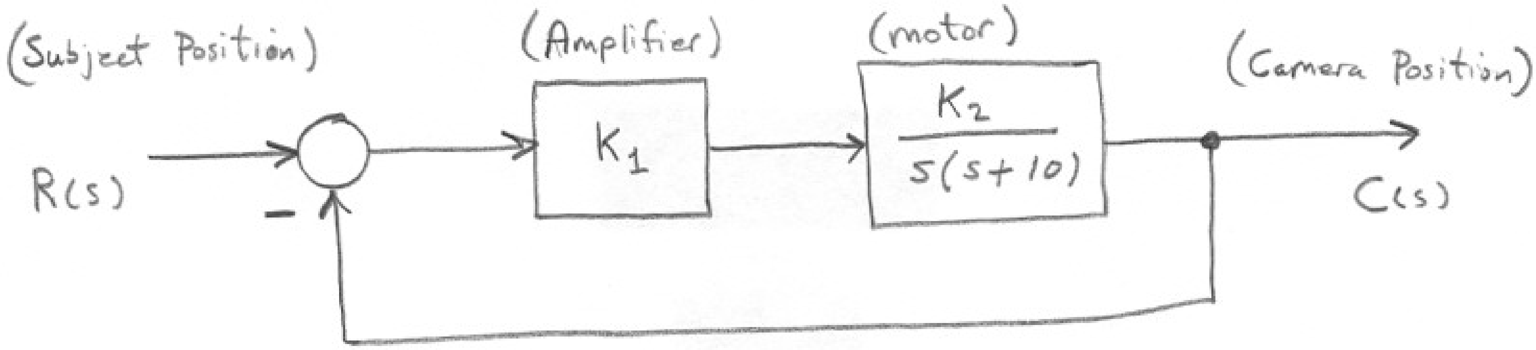
The closed loop transfer function is given by Eq. 1

$$\frac{C(s)}{R(s)} = \frac{K(s+1)(s+4)}{s^3 + (6+K)s^2 + (8+4K)s + 3K} \qquad (7)$$

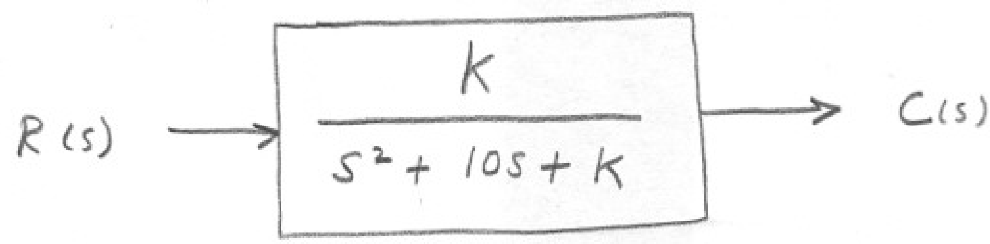
The poles of the closed loop transfer function are not immediately known w/out factoring the denominator and they are a function of K. There is no way of determining the system's performance except for specific values of k (unless we use a root locus plot).

The root locus plot will provide a picture of the closed loop pole locations as K is varied.

Now let's consider a first order plant and a proportional controller as shown below. The system is a model of a position tracking system



The closed loop transfer function is given by:



where $K = K_1 K_2$

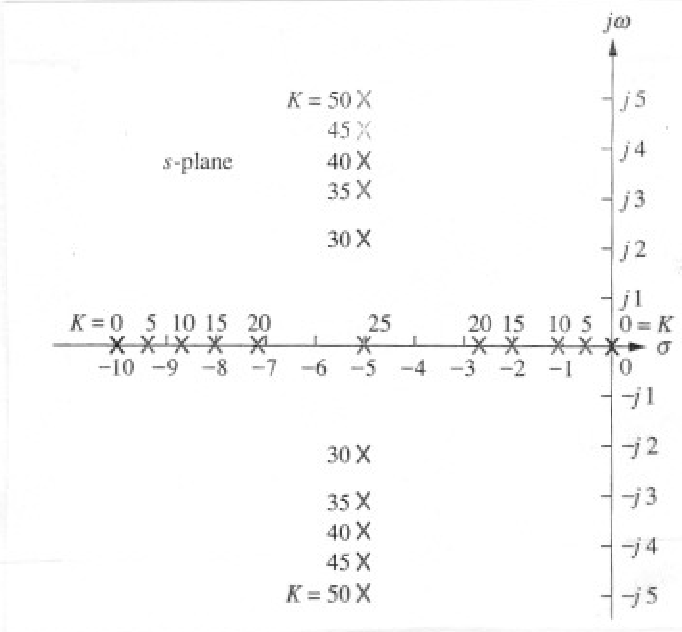
To determine the poles of the closed loop system we must apply the quadratic equation to the denominator of the closed loop transfer function

$$s_{1,2} = \frac{-10 \pm \sqrt{10^2 - 4K}}{2}$$

Table 8.1 Pole location as a function of gain for the system of Figure 8.4

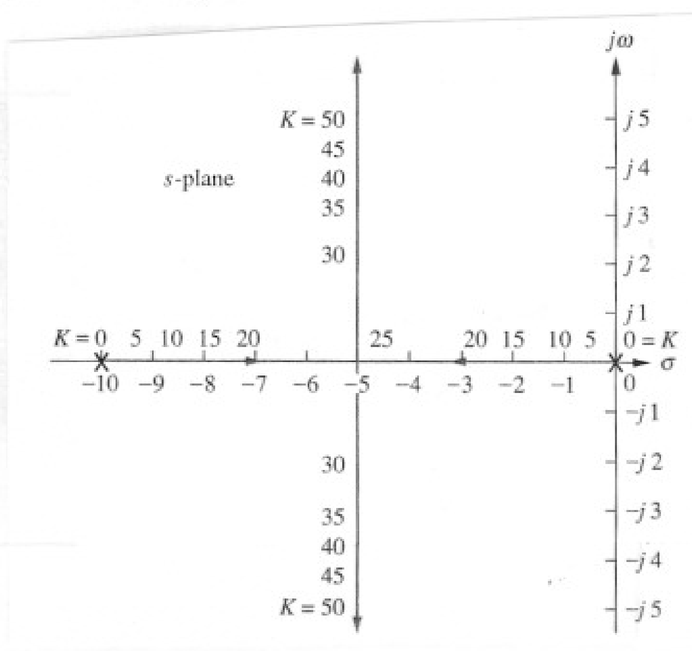
K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

These closed loop pole locations are displayed below as the gain K varies from 0 to 50. Notice how the poles move as the gain is changed.



As the gain increases, the closed loop pole which is at -10 for K=0 moves to the right and the pole which is at 0 for k=0 moves to the left. They meet at -5 and then break away from the real axis and move into the complex plane. If we replace the individual pole locations with solid lines, we obtain the root locus plot for the system.

The plot of the path of the closed loop poles as the gain is varied is what is called a "root locus".



- By looking at the root locus we can see that the poles are real for gains less than 25. This indicates that the system will have an overdamped response.
- For $K = 25$, the poles are real and repeated. This indicates the system is critically damped.
- For $K > 25$, the poles are complex conjugates. This indicates the system is underdamped.
- Also for $K > 25$ (underdamped) regardless of the gain, the real component of the complex poles are always the same. This indicates the settling time will remain the same as K is increased past 25. (For a second order system T_s is inversely proportional the real part of the complex pole).
- As K increases past 25 the damping ratio decreases while the percent overshoot increases. This results in a reduced peak time. (See p. 4-1)
- Lastly, the root locus never crosses into the right-half plane. Therefore the system always stable for any value of K!